OPERATIONAL CALCULUS FOR FUNCTIONS OF TWO INTEGER VARIABLES WITH SOME APPLICATIONS

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Operational calculus theory is developed for functions of two integer variables and applications given for solving some problems of discrete analysis.

The theory is developed in the present work for an operational calculus for functions of two integer variables; it is based on a discrete analog of the convolution that corresponds to the multiplication operation. Convolutions of similar type were previously considered in [1, 2]. Some new results are obtained.

Let S be the set of all either complex or real functions f(x, y) of two integer variables x and y which can assume all nonnegative integer values. The functions are denoted by either f(x, y), g(x, y), . . . or $a_{\nu\mu}$, $b_{\nu\mu}$, . . . , where x, y, ν, μ denote nonnegative integers. The set S is linear as regards standard operations of addition of functions and of multiplication by a number. The multiplication operation is now introduced in S.

Definition 1. The product of the functions $f(x, y) \in S$ and $g(x, y) \in S$ is called a function h(x, y) defined as follows:

$$\text{a) } h(0,\ 0)=f(0,\ 0)\ g(0,\ 0);$$

$$\text{b) } h(x,\ 0)=\sum_{\nu=0}^{x} f(x-\nu,\ 0)\,g(\nu,\ 0)-\sum_{\nu=0}^{x-1} f(x-1-\nu,\ 0)\,g(\nu,\ 0),$$

$$x=1,\ 2,\ 3,\ \ldots;$$

$$\text{c) } h(0,\ y)=\sum_{\mu=0}^{y} f(0,\ y-\mu)\,g(0,\ \mu)-\sum_{\mu=0}^{y-1} f(0,\ y-1-\mu)\,g(0,\ \mu),$$

$$y=1,\ 2,\ 3,\ \ldots;$$

$$\text{d) } h(x,\ y)=\sum_{\nu=0}^{x}\sum_{\mu=0}^{y} f(x-\nu,\ y-\mu)\,g(\nu,\ \mu)-\sum_{\nu=0}^{x-1}\sum_{\mu=0}^{y} f(x-1-\nu,\ y-\mu)\,g(\nu,\ \mu)$$
$$-\sum_{\nu=0}^{x}\sum_{\mu=0}^{y-1} f(x-\nu,\ y-1-\mu)\,g(\nu,\ \mu)+\sum_{\nu=0}^{x-1}\sum_{\mu=0}^{y-1} f(x-1-\nu,\ y-1-\mu)\,g(\nu,\ \mu),$$

$$x=1,\ 2,\ 3,\ \ldots;\ y=1,\ 2,\ 3,\ \ldots.$$

This product will be denoted by using the symbol "*". Thus

$$f(x,\ y)*g(x,\ y)=h(x,\ y). \tag{1}$$

The basic properties of this product are as follows.

1. The product has the commutative and associative property and also the distributive property with respect to addition.

2. If f(x, y) = c where c is constant then c * g(x, y) = cg(x, y).

3. If f(x, y) depends only on x, that is, f(x, y) = f(x) and g(x, y) depends only on y, that is, g(x, y) = g(y) then f(x, y) * g(x, y) = f(x)g(y), that is, the product in this case is identical with the ordinary product of functions. This property follows directly from the definition.

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The following can easily be shown.

4. If f(x, y) = f(x) and g(x, y) $\in$ S is an arbitrary function then the product f(x) $*$ g(x, y) = h(x, y) is such that

$$h(0, y) = f(0)\, g(0, y);\; y = 0, 1, 2, \ldots,$$

$$h(x, y) = \sum_{\nu=0}^{x} f(x-\nu)\, g(\nu, y) - \sum_{\nu=0}^{x-1} f(x-1-\nu)\, g(\nu, y);$$

$$x = 1, 2, 3, \ldots;\quad y = 0, 1, 2, \ldots$$

In a particular case the latter equality implies that

$$x * g(x, y) = \begin{cases} 0 & \text{for } x = 0;\; y = 0, 1, 2, \ldots, \\ \sum_{\nu=0}^{x-1} g(\nu, y) & \text{for } x = 1, 2, 3, \ldots;\; y = 0, 1, 2, \ldots \end{cases} \tag{2}$$

Similarly,

$$y * g(x, y) = \begin{cases} 0 & \text{for } y = 0;\; x = 0, 1, 2, \ldots, \\ \sum_{\mu=0}^{y-1} g(x, \mu) & \text{for } y = 1, 2, 3, \ldots;\; x = 0, 1, 2, \ldots \end{cases}$$

and

$$xy * g(x, y) = \begin{cases} \sum_{\nu=0}^{x-1} \sum_{\mu=0}^{y-1} g(\nu, \mu) & \text{for } x \geqslant 1,\; y \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

If into the set S one introduces side by side with ordinary addition of functions the product introduced in accordance with the formula (1) then S becomes a commutative ring. This ring has no divisors of zero [1, 2]. The extension of the ring S to the field of ratios is denoted by R(S), its elements being called operators. In particular, the operators 1/x and 1/y are elements of the field R(S). The notation $\sigma = 1/x$, $\tau = 1/y$ is introduced. It will now be explained what condition must be satisfied by f(x, y) $\in$ S so that the product $\sigma *$ f(x, y) is also an element of S. Let $\sigma *$ f(x, y) = h(x, y) $\in$ S then f(x, y) = 1/$\sigma *$ h(x, y) = x $*$ h(x, y). Hence it follows [see (2)] that f(0, y) = 0 for y = 0, 1, 2, . . . ; conversely, if this condition is satisfied then

$$f(x, y) = \sum_{\nu=0}^{x-1} [f(\nu+1, y) - f(\nu, y)].$$

By setting f(ν + 1, y) − f(ν, y) = h(ν, y) one obtains f(x, y) = $\sum_{\nu=0}^{x-1}$ h(ν, y). Therefore f(x, y) = x $*$ h(x, y), and hence h(x, y) = $\sigma *$ f(x, y) $\in$ S. One concludes similarly that the product $\tau *$ f(x, y) is an element of S only if f(x, 0) = 0 for x = 0, 1, 2, . . .

Thus the following theorem is valid.

THEOREM 1. For the product $\sigma *$ f(x, y) to be an element of the set S, f(0, y) = 0 for all y is a necessary and sufficient condition. In exactly the same way the condition f(x, 0) = 0 for all x is necessary and sufficient so that $\tau *$ f(x, y) $\in$ S.

Let us denote

$$f(x+1, y) - f(x, y) = \Delta_x f(x, y), \tag{3}$$

$$f(x, y+1) - f(x, y) = \Delta_y f(x, y), \tag{4}$$

Then

$$\sigma * [f(x, y) - f(0, y)] = \Delta_x f(x, y), \tag{5}$$

$$\tau * [f(x, y) - f(x, 0)] = \Delta_y f(x, y). \tag{6}$$

It follows from (3)-(6)

$$\sigma\tau * [f(x, y) - f(x, 0) - f(0, y) + f(0, 0)] = \Delta_x \Delta_y f(x, y).$$

If $f(x, 0) = f(0, y) = 0$ for all $x \geq 0$, $y \geq 0$ then $\sigma\tau * f(x, y) = \Delta_x\Delta_y f(x, y)$. One finds from (5) that

$$\sigma * [\Delta_x f(x, y) - \Delta_x f(0, y)] = \Delta_x^2 f(x, y),$$

and hence by again using (5) one obtains

$$\sigma \{\sigma [f(x, y) - f(0, y)] - \Delta_x f(0, y)\} = \Delta_x^2 f(x, y),$$

or

$$\Delta_x^2 f(x, y) = \sigma^2 f(x, y) - \sigma^2 f(0, y) - \sigma\Delta_x f(0, y). \tag{7}$$

Similarly,

$$\Delta_y^2 f(x, y) = \tau^2 f(x, y) - \tau^2 f(x, 0) - \tau\Delta_y f(x, 0). \tag{8}$$

It follows from the relations (7) and (8) that

$$(\Delta_x^2 + \Delta_y^2) f(x, y) = (\sigma^2 + \tau^2) f(x, y) - \sigma^2 f(0, y) - \tau^2 f(x, 0) - \sigma\Delta_x f(0, y) - \tau\Delta_y f(x, 0).$$

Let

$$x^{(n)} = x(x-1)(x-2)\ \ldots\ (x-n+1),\ x^{(0)} = 1,$$
$$y^{(m)} = y(y-1)(y-2)\ \ldots\ (y-m+1),\ y^{(0)} = 1.$$

Since $\Delta_x x^{(n)} = nx^{(n-1)}$ then in view of $x^{(n)}|_{x=0} = 0$ for $n > 0$, one finds from (5) that $\sigma x^{(n)} = \Delta_x x^{(n)} = nx^{(n-1)}$, hence $\sigma^n x^{(n)} = n!$ or $1/\sigma^n = x^{(n)}/n!$. It is obvious that $1/\tau^m = y^{(m)}/m!$.

Definition 2. Let

$$\eta_{m,n}(x, y) = \begin{cases} 1; & x \geqslant m,\ y \geqslant n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\eta_k(x) = \begin{cases} 1; & x \geqslant k, \\ 0; & x < k,\ \eta_0(x) \equiv 1. \end{cases}$$

Obviously $\eta_{m,n}(x, y) = \eta_m(x)\eta_n(y)$ then (see Section 3) $\eta_{m,n}(x, y) = \eta_m(x) * \eta_n(y)$. It is easily verified that

$$\eta_m(x) * \eta_{m_1}(x) = \eta_{m+m_1}(x), \tag{9}$$

and therefore

$$\eta_{m,n}(x, y) * \eta_{m_1,n_1}(x, y) = \eta_{m+m_1,n+n_1}(x, y).$$

moreover,

$$(1+\sigma)\eta_1(x) = \eta_1(x) + \sigma\eta_1(x) = \eta_1(x) + \Delta_x\eta_1(x) = \eta_1(1+x).$$

But $\eta_1(x+1) \equiv 1$, $x = 0, 1, 2, \ldots,$ and therefore

$$\eta_1(x) = \frac{1}{1+\sigma}. \tag{10}$$

By using (9) it is found that $\eta_m(x) = \eta_1^m(x)$ and consequently

$$\eta_m(x) = \frac{1}{(1+\sigma)^m}.$$

Similarly,

$$\eta_n(y) = \frac{1}{(1+\tau)^n},$$

and consequently

$$\eta_{m,n}(x, y) = \frac{1}{(1+\sigma)^m(1+\tau)^n}.$$

Let us consider the double series

$$\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{f(\nu, \mu)}{(1+\sigma)^\nu(1+\tau)^\mu} = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} f(\nu, \mu)\left[\left(\frac{1}{(1+\sigma)^\nu} - \frac{1}{(1+\sigma)^{\nu+1}}\right)\left(\frac{1}{(1+\tau)^\mu} - \frac{1}{(1+\tau)^{\mu+1}}\right)\right]$$

$$=\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} f(\nu,\ \mu)\left[\eta_{\nu,\mu}(x,\ y)-\eta_{\nu+1,\mu}(x,\ y)-\eta_{\nu,\mu+1}(x,\ y)+\eta_{\nu+1,\mu+1}(x,\ y)\right]=f(x,\ y).$$

Thus

$$\bar{f}(\sigma,\ \tau)=\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{f(\nu,\ \mu)}{(1+\sigma)^{\nu}(1+\tau)^{\mu}}=f(x,\ y). \tag{11}$$

A number of formulas will now be obtained which are analogs of the corresponding formulas of the operational calculus in two variables [3].

1. Let

$$f(x,\ y)=\begin{cases}0, & x\neq y,\\ f(x) & x=y,\end{cases}$$

where f(x) is a given function of an integer argument. Then

$$f(x,\ y)=\bar{f}(\sigma,\ \tau)=\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\sum_{\nu=0}^{\infty}\frac{f(\nu)}{(1+\sigma)^{\nu}(1+\tau)^{\nu}}=\frac{\sigma\tau}{1+\sigma+\tau+\sigma\tau}\sum_{\nu=0}^{\infty}\frac{f(\nu)}{(1+\sigma+\tau+\sigma\tau)^{\nu}},$$

or by setting

$$\bar{f}(\sigma)=\frac{\sigma}{1+\sigma}\sum_{\nu=0}^{\infty}\frac{f(\nu)}{(1+\sigma)^{\nu}},$$

it is found that

$$\frac{\sigma\tau}{\sigma+\tau+\sigma\tau}\bar{f}(\sigma+\tau+\sigma\tau)=\begin{cases}0, & x\neq y,\\ f(x), & x=y.\end{cases} \tag{12}$$

In particular, for f(x) ≡ 1 one obtains from (12)

$$\frac{\sigma\tau}{\sigma+\tau+\sigma\tau}=\begin{Bmatrix}0, & x\neq y\\ 1, & x=y\end{Bmatrix}=\delta_{xy}. \tag{13}$$

2. Let m = min(x, y) and f(x) be a given function. We set f(x, y) = f(m). One obtains

$$f(m)=\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\left\{\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu}\frac{f(\mu)}{(1+\sigma)^{\nu}(1+\tau)^{\mu}}\right.$$

$$\left.+\sum_{\nu=0}^{\infty}\sum_{\mu=\nu+1}^{\infty}\frac{f(\nu)}{(1+\sigma)^{\nu}(1+\tau)^{\mu}}\right\}=\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\left\{\sum_{\mu=0}^{\infty}\frac{f(\mu)}{(1+\sigma)^{\mu}(1+\tau)^{\mu}}\right.$$

$$\left.+\sum_{\mu=0}^{\infty}\frac{f(\mu)}{\sigma(1+\sigma)^{\mu}(1+\tau)^{\mu}}+\sum_{\nu=0}^{\infty}\frac{f(\nu)}{\tau(1+\sigma)^{\nu}(1+\tau)^{\nu}}\right\}=\frac{\sigma+\tau+\sigma\tau}{1+\sigma+\tau+\sigma\tau}\sum_{\nu=0}^{\infty}\frac{f(\nu)}{(1+\sigma+\tau+\sigma\tau)^{\nu}}.$$

By setting

$$\bar{f}(\sigma)=\frac{\sigma}{1+\sigma}\sum_{\nu=0}^{\infty}\frac{f(\nu)}{(1+\sigma)^{\nu}},$$

one obtains

$$\bar{f}(\sigma+\tau+\sigma\tau)=f(m).$$

3. Let f(x, y) = f(x + y). Then

$$f(x+y)=\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{f(\nu+\mu)}{(1+\sigma)^{\nu}(1+\tau)^{\mu}}=\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\sum_{l=0}^{\infty}f(l)\sum_{\nu+\mu=l}\frac{1}{(1+\sigma)^{\nu}(1+\tau)^{\mu}}$$

$$=\frac{\sigma\tau}{\tau-\sigma}\sum_{l=0}^{\infty}f(l)\left[\frac{1}{(1+\sigma)^{l+1}}-\frac{1}{(1+\tau)^{l+1}}\right].$$

By setting

$$\bar{f}(\sigma) = \frac{\sigma}{1+\sigma} \sum_{\nu=0}^{\infty} \frac{f(\nu)}{(1+\sigma)^{\nu}},$$

one finds that

$$\frac{\sigma \bar{f}(\tau) - \tau \bar{f}(\sigma)}{\sigma - \tau} = f(x+y).$$

Further two functional relations will be given which may prove useful in obtaining the operators. Let $\bar{f}(\sigma, \tau) = f(x, y)$, $\bar{g}(\sigma, \tau) = g(x, y)$; then [see (11)]

$$\frac{(1+\sigma)(1+\tau)}{\sigma\tau} \bar{f}(\sigma, \tau) \bar{g}(\sigma, \tau) = \frac{\sigma\tau}{(1+\sigma)(1+\tau)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(1+\sigma)^n (1+\tau)^m} \sum_{p=0}^{n} \sum_{q=0}^{m} f(n-p, m-q)\, g(p, q).$$

Hence it follows that

$$\frac{(1+\sigma)(1+\tau)}{\sigma\tau} \bar{f}(\sigma, \tau) \bar{g}(\sigma, \tau) = \sum_{p=0}^{x} \sum_{q=0}^{y} f(x-p, y-q)\, g(p, q).$$

One can establish in a similar manner that

$$\frac{\bar{f}(\sigma, \tau) \bar{g}(\sigma, \tau)}{\sigma\tau} = \begin{cases} \sum_{\nu=0}^{x-1} \sum_{\mu=0}^{y-1} f(x-\nu-1, y-1-\mu)\, g(\nu, \mu), & x \geqslant 1,\ y \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

By setting $f(x, y) = (1+\alpha)^x$ and $f(x, y) = (1+\beta)^y$ in the formulas (5)-(6) one finds

$$\frac{\sigma}{\sigma-\alpha} = (1+\alpha)^x, \quad \frac{\tau}{\tau-\beta} = (1+\beta)^y.$$

By differentiating the above relations with respect to α and β respectively one obtains

$$\frac{\sigma}{(\sigma-\alpha)^{n+1}} = \frac{x^{(n)}}{n!} (1+\alpha)^{x-n}, \quad \frac{\tau}{(\tau-\beta)^{n+1}} = \frac{y^{(n)}}{n!} (1+\beta)^{y-n}.$$

By setting $f(x) = \lambda^x/x!$ in (11) one obtains

$$\bar{f}(\sigma) = \frac{\sigma}{\sigma+1} \exp\left(\frac{\lambda}{1+\sigma}\right) = \frac{\lambda^x}{x!}.$$

Hence it follows that

$$\frac{\sigma\tau}{(1+\sigma)(1+\tau)} \exp\left(-\frac{\lambda\sigma}{1+\sigma} - \frac{\mu\tau}{1+\tau}\right) = \frac{\lambda^x \mu^y}{x!\, y!} e^{-\lambda-\mu}. \tag{14}$$

If one assumes that the integrals are convergent one obtains from (14)

$$\frac{\sigma\tau}{(1+\sigma)(1+\tau)} \int_0^{\infty} \int_0^{\infty} \exp\left(-\frac{\lambda\sigma}{1+\sigma} - \frac{\mu\tau}{1+\tau}\right) \Phi(\lambda, \mu)\, d\lambda d\mu = \frac{1}{x! y!} \int_0^{\infty} \int_0^{\infty} \lambda^x \mu^y e^{-\lambda-\mu} \Phi(\lambda, \mu)\, d\lambda d\mu. \tag{15}$$

If one sets $\Phi(\lambda, \mu) = \Phi(\lambda, \mu)$ in (15) then the left-hand side of the equality becomes

$$\frac{2\sigma\tau}{(1+\sigma)(1+\tau)} \int_0^{\infty} \Phi(\xi) K_0 \left[2 \sqrt{\frac{\sigma\tau\xi}{(1+\sigma)(1+\tau)}}\right] d\xi,$$

and the right-hand side becomes

$$\frac{2}{x!\, y!} \int_0^{\infty} \xi^{\frac{x+y}{2}} K_{x-y} (2\sqrt{\xi})\, \Phi(\xi)\, d\xi,$$

and in both cases use has been made of the equality

$$\int_0^{\infty} x^{\nu-1} \exp\left(-\gamma x - \frac{\beta}{x}\right) dx = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_{\nu} (2\sqrt{\beta\gamma}).$$

Thus,

$$\frac{\sigma\tau}{(1+\sigma)(1+\tau)}\int_0^\infty \Phi(\xi) K_0\left[2\sqrt{\frac{\sigma\tau\xi}{(1+\sigma)(1+\tau)}}\right] d\xi = \frac{1}{x!\,y!}\int_0^\infty \Phi(\xi)\,\xi^{\frac{x+y}{2}} K_{x-y}(2\sqrt{\xi})\,d\xi \quad (16)$$

or

$$\int_0^\infty \Phi\left[\frac{(1+\sigma)(1+\tau)}{\sigma\tau}\,t\right] K_0(2\sqrt{t})\,dt = \frac{1}{x!\,y!}\int_0^\infty \Phi(\xi)\,\xi^{\frac{x+y}{2}} K_{x-y}(2\sqrt{\xi})\,d\xi.$$

The formulas (15) and (16) can be employed to construct the tables of the values of the operators $\bar{f}(\sigma, \tau)$. Of course,

$$\exp\left(-\frac{\xi}{\sigma}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!}\,\frac{1}{\sigma^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!}\,\frac{x^{(n)}}{n!}.$$

Bearing in mind that x is an integer and that $x^{(n)} = 0$ for $n > x$ one obtains

$$\exp\left(-\frac{\xi}{\sigma}\right) = L_x(\xi), \quad (17)$$

and in a similar manner

$$\exp\left(-\frac{\eta}{\tau}\right) = L_y(\eta),$$

where $L_x(\xi)$ is the Laguerre polynomial of degree x. It is known [4] that these polynomials form an orthonormal system on the interval $(0, \infty)$ with weight $e^{-\xi}$. This property of the polynomials follows directly from the formula (13). Indeed,

$$\int_0^\infty L_x(\xi) L_y(\xi) e^{-\xi} d\xi = \int_0^\infty \exp\left(-\frac{\xi}{\sigma} - \frac{\xi}{\tau} - \xi\right) d\xi = \frac{\sigma\tau}{\sigma+\tau+\sigma\tau} = \begin{cases} 0, & x \neq y, \\ 1, & x = y. \end{cases}$$

Note. Here it is necessary to use the property (3) of the product in the ring S. From (17) one has

$$\int_0^\infty\int_0^\infty \exp\left(-\frac{\xi}{\sigma} - \frac{\eta}{\tau}\right) f(\xi, \eta)\,d\xi d\eta = F\left(\frac{1}{\sigma}, \frac{1}{\tau}\right).$$

In the above

$$F(p, q) = \int_0^\infty\int_0^\infty f(\xi, \eta)\,e^{-p\xi - q\eta}\,d\xi d\eta.$$

By using the available tables for the two-dimensional Laplace transformation one can find the operator $F(1/\sigma, 1/\tau)$, namely

$$F\left(\frac{1}{\sigma}, \frac{1}{\tau}\right) = \int_0^\infty\int_0^\infty L_x(\xi) L_y(\eta) f(\xi, \eta)\,d\xi d\eta. \quad (18)$$

It is not difficult to establish that

$$\frac{1}{\sigma^r}\exp\left(-\frac{\xi}{\sigma}\right) = L_x^r(\xi), \quad \frac{1}{\tau^s}\exp\left(-\frac{\eta}{\tau}\right) = L_y^s(\eta);$$

$$\int_0^\infty L_x^r(\xi) L_y^r(\xi)\,\xi^r e^{-\xi} d\xi = \int_0^\infty \frac{1}{\sigma^r}\exp\left(-\frac{\xi}{\sigma}\right)\frac{1}{\tau^r}\exp\left(-\frac{\xi}{\tau}\right)\xi^r e^{-\xi} d\xi = \frac{\Gamma(r+1)\,\sigma r}{(\sigma+\tau+\sigma\tau)^{r+1}} = \begin{cases} 0, & x \neq y, \\ \Gamma(1+r)\,x^{(r)}, & x = y. \end{cases}$$

The equality (18) now becomes

$$\frac{1}{\sigma^r}\frac{1}{\tau^s} F\left(\frac{1}{\sigma}, \frac{1}{\tau}\right) = \int_0^\infty\int_0^\infty L_x^r(\xi) L_y^s(\eta) f(\xi, \eta)\,d\xi d\eta.$$

Another method for finding the value of $\bar{f}(\sigma, \tau)$ consists in what follows. Let, as usual, $|\xi|$ denote the integral part of a number ξ. If $|\xi| = \nu$ and $|\eta| = \mu$ then

$$f([\xi], [\eta]) = f(\nu, \mu) \quad \text{for} \quad \nu \leqslant \xi < \nu + 1, \ \mu \leqslant \eta < \mu + 1.$$

We shall now find the Laplace – Carson transformation for the function $f([\xi], [\eta])$. One obtains

$$F(p, q) = pq \int_0^\infty \int_0^\infty f([\xi], [\eta]) e^{-p\xi - q\eta} d\xi d\eta = pq \sum_{\nu=0}^\infty \sum_{\mu=0}^\infty f(\nu, \mu) \int_\nu^{\nu+1} \int_\mu^{\mu+1} e^{-p\xi - q\eta} d\xi d\eta$$

$$= (e^p - 1)(e^q - 1) \sum_{\nu=0}^\infty \sum_{\mu=0}^\infty f(\nu, \mu) e^{-p(\nu+1)} e^{-q(\mu+1)}.$$

But [see (10)]

$$e^{-p} = \eta_1(x) = \frac{1}{1+\sigma}, \quad e^{-q} = \eta_1(y) = \frac{1}{1+\tau}, \tag{19}$$

therefore

$$F(p, q) = \frac{\sigma\tau}{(1+\sigma)(1+\tau)} \sum_{\nu=0}^\infty \sum_{\mu=0}^\infty \frac{f(\nu, \mu)}{(1+\sigma)^\nu (1+\tau)^\mu} = \bar{f}(\sigma, \tau).$$

The latter formula enables one to find the operators $\bar{f}(\sigma, \tau)$ by using the table of the two-dimensional Laplace transformation and the equalities (19). Let us consider another example. Let

$$2^{\frac{\xi}{2}} \frac{(1+\sigma-\sqrt{2})^\xi \sigma}{(\sqrt{2}+\sigma\sqrt{2}-1)^{\xi+1}} = l_x(\xi).$$

One obtains

$$\sum_{j=0}^\infty l_x(j) l_y(j) 2^{-j} = \sigma\tau \sum_{k=0}^\infty \frac{(1+\sigma-\sqrt{2})^k (1+\tau-\sqrt{2})^k}{(\sqrt{2}+\sigma\sqrt{2}-1)^{k+1}(\sqrt{2}+\tau\sqrt{2}-1)^{k+1}} = \frac{\sigma\tau}{(\sqrt{2}+\sigma\sqrt{2}-1)(\sqrt{2}+\tau\sqrt{2}-1)}$$

$$\times \frac{1}{1 - \dfrac{(1+\sigma-\sqrt{2})(1+\tau-\sqrt{2})}{(\sqrt{2}+\sigma\sqrt{2}-1)(\sqrt{2}-\tau\sqrt{2}-1)}} = \frac{\sigma\tau}{\sigma+\tau+\sigma\tau} = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Therefore if one takes as a scalar product

$$(l_n(\xi), l_m(\xi)) = \sum_{j=0}^\infty l_n(j) l_m(j) 2^{-j},$$

then the system of polynomials $l_0(\xi), l_1(\xi), \ldots, l_n(\xi)$ is an orthonormal system. We shall now find an explicit expression for $l_n(\xi)$. One has

$$l_x(\xi) = \frac{\sigma 2^{\frac{\xi}{2}} (\sigma+1-\sqrt{2})^\xi}{2^{\frac{\xi+1}{2}} \left(\sigma+1-\dfrac{1}{\sqrt{2}}\right)^{\xi+1}},$$

but

$$\frac{\sigma+1-\sqrt{2}}{\sigma+1-\dfrac{1}{\sqrt{2}}} = \frac{\sigma+1-\dfrac{1}{\sqrt{2}} - \left(\sqrt{2}-\dfrac{1}{\sqrt{2}}\right)}{\sigma+1-\dfrac{1}{\sqrt{2}}} = \left(1 - \frac{\sqrt{2}-\dfrac{1}{\sqrt{2}}}{\sigma+1-\dfrac{1}{\sqrt{2}}}\right)^\xi \frac{\sigma}{\sqrt{2}\left(\sigma+1-\dfrac{1}{\sqrt{2}}\right)}.$$

Consequently,

$$l_x(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^\xi (-1)^k \binom{\xi}{k} \frac{\left(\dfrac{1}{\sqrt{2}}\right)^k \sigma}{\left(\sigma+1-\dfrac{1}{\sqrt{2}}\right)^{k+1}},$$

but

$$\frac{\sigma}{(\sigma-\alpha)^{k+1}}=\frac{x^{(k)}}{k!}(1+\alpha)^{x-k},$$

and therefore

$$\frac{\sigma}{\left(\sigma+1-\frac{1}{\sqrt{2}}\right)^{k+1}}=\frac{x^{(k)}}{k!}\left(1-1+\frac{1}{\sqrt{2}}\right)^{x-k}.$$

Thus

$$l_x(\xi)=\frac{1}{\sqrt{2}}\sum_{k=0}^{\xi}(-1)^k\binom{\xi}{k}\frac{x^{(k)}}{k!}\left(\frac{1}{\sqrt{2}}\right)^x$$

or

$$l_x(\xi)=2^{-\frac{x+1}{2}}\sum_{k=0}^{\xi}(-1)^k\frac{\xi^{(k)}}{k!}\frac{x^{(k)}}{k!}.$$

It follows from the above that the polynomials [5, 6]

$$l_n(\xi)=2^{-\frac{n+1}{2}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\xi^{(k)}}{k!}$$

satisfy the relation

$$\sum_{j=0}^{\infty}l_n(j)\,l_m(j)\,2^{-j}=\begin{cases}1, & m=n,\\ 0, & m\neq n.\end{cases}$$

In conclusion, a generalization is given of a formula encountered in the summation of series [7] for the case of a function of two variables. It will now be proved that

$$\sum_{\nu=0}^{[x-1]}\sum_{\mu=0}^{[y-1]}f(\nu,\mu)(-1)^{\nu}(-1)^{\mu}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{2^{n+m+2}}\Delta_x^n\Delta_y^m f(0,0)-(-1)^{[x]}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{2^{n+m+2}}\Delta_x^n\Delta_y^m f[x,0]$$

$$-(-1)^{[y]}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{2^{n+m+2}}\Delta_x^n\Delta_y^m f[0,y]+(-1)^{[x]+[y]}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{2^{n+m+2}}\Delta_x^n\Delta_y^m f[x,y]. \tag{20}$$

The corresponding formula in the case of a single variable is

$$\sum_{k=0}^{[t-1]}f(k)(-1)^k=\sum_{n=0}^{\infty}\frac{(-1)^n\Delta^n f(0)}{2^{n+1}}-(-1)^{[t]}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\Delta^n f[t]. \tag{21}$$

The above formula follows from the formula (see [8])

$$\frac{1}{r}f[t]*g[t]=\sum_{k=0}^{[t-1]}f[t-1-k]\,g(k),\quad t\geqslant 1,\ r=e^p-1. \tag{22}$$

By setting $g[t]=(-1)^{[t]}=r/(r+2)$ one obtains

$$\frac{1}{r}f[t]*g[t]=\frac{1}{r+2}f[t]$$

and consequently

$$\frac{1}{r+2}f[t]=\sum_{k=0}^{[t-1]}f[t-1-k](-1)^k=\sum_{k=0}^{[t-1]}(-1)^{[t-1-k]}f(k)=(-1)^{[t-1]}\sum_{k=0}^{[t-1]}f(k)(-1)^k.$$

From the latter relation the equality (21) follows together with the formula

$$\frac{1}{r+2}f[t]=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\{\Delta^n f[t]-(-1)^{[t]}\Delta^n f(0)\}$$

Using (21) in the variable y one obtains

$$\sum_{\mu=0}^{[y-1]} f[x, \mu](-1)^{\mu} = \sum_{m=0}^{\infty} \frac{(-1)^m \Delta_y^m f[x, 0]}{2^{m+1}} - (-1)^{[y]} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta_y^m f[x, y]}{2^{m+1}}.$$

By applying the same formula to the variable x and the function

$$\Phi[x] = \sum_{\mu=0}^{[y-1]} f[x, \mu](-1)^{\mu},$$

one obtains

$$\sum_{\nu=0}^{[x-1]} \sum_{\mu=0}^{[y-1]} f(\nu, \mu)(-1)^{\nu}(-1)^{\mu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \Delta_x^n \sum_{\mu=0}^{[y-1]} f[0, \mu](-1)^{\mu} - (-1)^{[x]} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \Delta_x^n \sum_{\mu=0}^{[y-1]} f[x, \mu](-1)^{\mu}.$$

The required formula (20) is obtained by replacing in the right-hand side of the above equation the sums $\sum_{\mu=0}^{[y-1]}$ by using the formula (22).

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